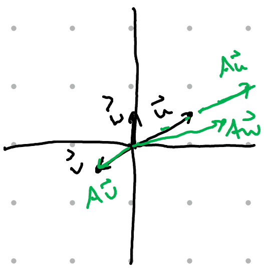
5.1: Eigenvectors and eigenvalues

Key idea: Associated to many matrices are special vectors that simply describe the action of multiplying by said matrix. These vectors ease computation and increase understanding of a linear transformation.

Today we consider the eigenvectors of a matrix. These are vectors which are incredibly well-behaved under the transformation defined by the matrix.

Ex Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{u}.$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (1)\begin{bmatrix} -1 \\ -1 \end{bmatrix} = 1 \cdot \vec{v}.$$

$$A\vec{w} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for any } \lambda.$$

Notice A sends $\vec{u}$ and $\vec{v}$ to scalar multiples of $\vec{u}$ and $\vec{v}$. This is the behavior we seek.

Def: An *eigenvector* of an $n \times n$ matrix A is any nonzero vector $\vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is an *eigenvalue* of A if there is a nontrivial solution to $A\vec{x} = \lambda\vec{x}$. We say $\vec{x}$ is an *eigenvector with eigenvalue λ*.

Ex $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2$, $\lambda_1 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (resp.)

Ex Show that 7 is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find all eigenvectors with this eigenvalue.

Note 7 is an eigenvalue if $A\vec{x} = 7\vec{x}$ has a nontrivial solution

$$\Rightarrow (A\vec{x} - 7\vec{x}) = \vec{0} \Rightarrow \underbrace{(A - 7I_2)\vec{x} = \vec{0}}_{\text{homogeneous equation}}. \Rightarrow \left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So any nontrivial solution to this equation is an eigenvector:

$$\underbrace{\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}}_{(A-7I)} \vec{x} = \vec{0} \quad \rightsquigarrow \begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_1 = x_2$, x_2 free,

So, if $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for any nonzero x_2 with eigenvalue 7.

$\left(\text{e.g. } A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

Note we can show in the same way that -4 is an eigen value of A with corresponding eigenvalues $\vec{x} = t\begin{bmatrix} 6 \\ -5 \end{bmatrix}$.

In each case we see an eigenvalue has many eigenvectors.
In fact, these eigenvectors form a subspace.

Def: If λ is an eigenvalue of A ($n \times n$) then the collection of all eigenvectors with eigenvalue λ (and $\vec{0}$) is the eigenspace (of A corresponding to λ)

Fact: The eigenspace for some λ is a subspace of $\mathbb{R}^n$.

Why? The eigenspace for λ is $\text{Nul}(A - \lambda I)$
$= \{ \vec{x} : (A - \lambda I)\vec{x} = \vec{0}$
$\hookrightarrow A\vec{x} = \lambda\vec{x} \}$.

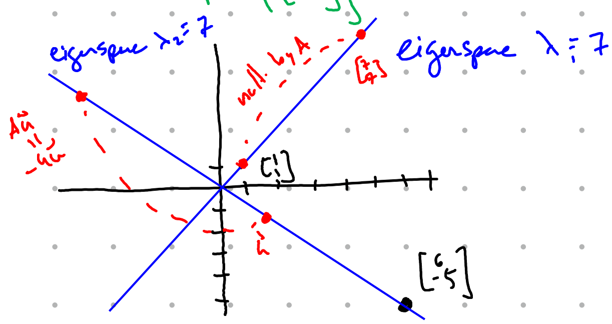
Ex| Sketch the eigenspace of $\lambda_1 = 7$, $\lambda_2 = -4$ for $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

E-space $\lambda_1 = 7$: $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

All e-vectors for $\lambda_1 = 7$ are of the form $\vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

E-space $\lambda_2 = -4$: $\text{Span}\left\{ \begin{bmatrix} 6 \\ -5 \end{bmatrix} \right\}$

All e-vectors for $\lambda_2 = -4$ are of the form $\vec{x} = t\begin{bmatrix} 6 \\ -5 \end{bmatrix}$.



Ex Let $A=\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}$. Note A has an eigenvalue 2. Find a basis for the eigenspace.

So we need a basis for $\mathrm{Nul}((A-2I))$.

$$A-2I=\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}-2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}-\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}2 & -1 & 6\\ 2 & -1 & 6\\ 2 & -1 & 6\end{bmatrix}$$

And find all $\vec{x}$ s.t. $(A-2I)\vec{x}=\vec{0}$

$$\begin{bmatrix}2 & -1 & 6 & 0\\ 2 & -1 & 6 & 0\\ 2 & -1 & 6 & 0\end{bmatrix}\sim\begin{bmatrix}2 & -1 & 6 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -1/2 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\Rightarrow \vec{x}=x_2\begin{bmatrix}1/2\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}.$$

So a basis is $\left\{\begin{bmatrix}1\\ 2\\ 0\end{bmatrix},\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}\right\}$.

Consider "a 2D eigenspace" on the webpage

Fact: If $\{\vec{v}_1,\dots,\vec{v}_p\}$ are eigenvectors of A with distinct eigenvalues $\lambda_1,\dots,\lambda_p$ then $\{\vec{v}_1,\dots,\vec{v}_p\}$ is linearly independent.

For later: we'll see this fact and it's use in section 5.3

As a preview, note we can determine all we want about the eigenvectors of A after knowing an eigenvalue. So, ..., how do we find an eigenvalue of A?

Need λ s.t.

$A\vec{x}=\lambda\vec{x}$ has nontrivial solution $\Rightarrow (A-\lambda I)\vec{x}=\vec{0}$ has nontrivial solution $\Rightarrow (A-\lambda I)$ is not invertible $\Rightarrow \det(A-\lambda I)=0$

characteristic equation.

e.g. first $A=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$ $\rightsquigarrow$ $\det(A-\lambda I)=0$

$\lambda^2-3\lambda+2=0 \Rightarrow (\lambda-2)(\lambda-1)=0$

$\Rightarrow \lambda=2,\ \lambda=1.$